7) Introduction to scattering

I) Classification of scattering problems

Scattering = open systems = systems coupled to external propagating modes

= we probe the system from the exterior (incident field)

currents are induced inside the system and radiate out (scattered field)

incident field + scattered field = total field (the one satisfying ME + BCs)

* 2 possibilities after fixing the frequency

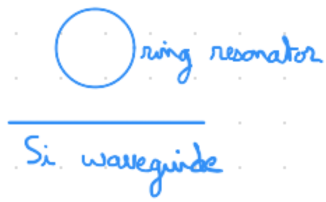
1) We couple to the system via a finite number of ports $m \in \mathbb{N}$

$\boxed{\text{m-port scattering}}$

1 port = 1 mode carrying forward and backward waves

ex: electrical networks & transmission lines

dielectric resonators & optical fibers



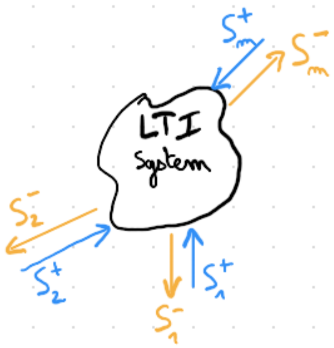
2) We are in free space (infinite number of modes = radiation continuum)

$\boxed{\text{free space scattering}}$

ex: radars, scattering of sunlight by small particles in the sky.

II) Linear M-port scattering

1. The scattering matrix



LTI : linear time-invariant,

$S_i^+ \in \mathbb{C}$: incident wave amplitude at port $i \in [\![1, m]\!]$

$S_i^- \in \mathbb{C}$: outgoing — ——— ————

Convention : $S_i^\pm$ is normalized such that $|S_i^\pm|^2 = S_i^{\pm\dagger} S_i^\pm$ is the incident/outgoing power at port i.

Linearity $\Rightarrow$ $\exists\, S$ / $\boxed{\vec{S}_- = S\, \vec{S}_+}$ $m \times m$ scattering matrix

$\vec{S}_\pm = [S_1^\pm, S_2^\pm, \dots, S_m^\pm]^T$ $m \times 1$ column vector

2. Energy conservation

2. $\vec{S}_+^\dagger \cdot \vec{S}_+ = \sum_{i=1}^{m} S_i^{+*} S_i^{+} = \sum_{i=1}^{m} |S_i^+|^2$ total incident power

$\vec{S}_-^\dagger \vec{S}_- = \sum_{i=1}^{m} |S_i^-|^2$ total outgoing power

Energy conservation $\Leftrightarrow \vec{S}_+^\dagger \vec{S}_+ = \vec{S}_-^\dagger \vec{S}_- \quad \forall \vec{S}_+$

$\Leftrightarrow \vec{S}_+^\dagger \vec{S}_+ = (S\vec{S}_+)^\dagger (S\vec{S}_+) = \vec{S}_+^\dagger S^\dagger S \vec{S}_+ \quad \forall \vec{S}_+$

$\Leftrightarrow S^\dagger S = \mathbb{1}$, S is unitary

$$\boxed{S \text{ conserves the power flux} \Leftrightarrow S \text{ is unitary}}$$

In particular: $\left\{\begin{array}{l}\text{losses are negligible} \\ \text{gain \& loss compensate}\end{array}\right. \Rightarrow S$ is unitary

Some useful mathematical properties of unitary matrices:

1. $S^\dagger S = \mathbb{1} = S S^\dagger$
2. $S^{-1} = S^\dagger$
3. $|\det S| = 1$
4. $\exists$ an orthonormal basis of eigenvectors of S with eigenvalues lying on the unit circle
5. S has a decomposition of the form $S = VDV^\dagger$ with V unitary and D is diagonal unitary.
6. $\exists$ H Hermitian such that $S = e^{iH}$ (matrix exponential)

2. General parametrization of 2×2 unitary matrices: let $a, b \in \mathbb{C}$, $|a|^2 + |b|^2 = 1$

$$U = \begin{pmatrix} a & b \\ -e^{j\varphi} b^* & e^{j\varphi} a^* \end{pmatrix}, \text{ with } \det U = e^{j\varphi} \Rightarrow 4 \text{ parameters}$$

For unitary scattering, we write $a = r e^{j\alpha}$, $b = t e^{j\beta}$ with $r = \cos\theta$, $t = \sin\theta$. Then we get:

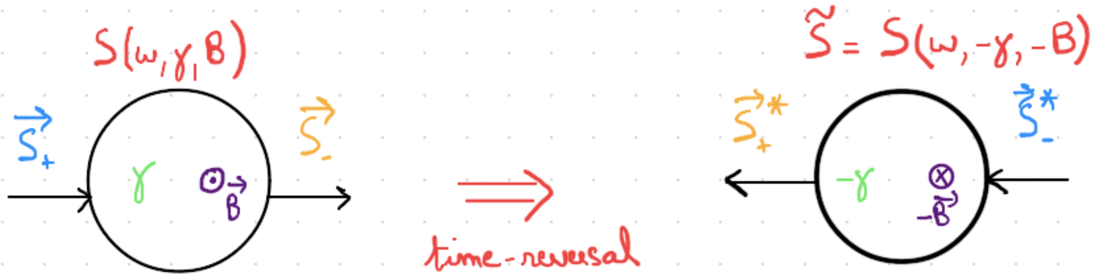
$$\boxed{S = \begin{pmatrix} r e^{j\alpha} & t e^{j\beta} \\ -e^{j\varphi} t e^{-j\beta} & e^{j\varphi} r e^{-j\alpha} \end{pmatrix}}$$

3. Time-reversal symmetry

3. Consider a general m-port scattering process described by $S(\omega, \gamma, B)$ where $\omega \in \mathbb{R}$ is the frequency and (γ, B) are two different sources of broken time-reversal symmetry:

* γ describes absorption or amplification ($\gamma > 0$ or $\gamma < 0$), which contributes to an imaginary symmetric term to the Hamiltonian (non-Hermitian) or to the non-unitarity of S

* B describes a time-odd parameter (e.g. magneto-optical effect, Aharonov Bohm phase, etc) that contributes to an imaginary anti-symmetric term to the Hamiltonian, i.e does not break unitarity of S.



NB: The effect of time reversal on the complex signal amplitudes $\vec{S}_{\pm}$ is complex conjugation since $t \to -t$ effectively conjugate the temporal exponent $e^{j\omega t}$ and reverse the propagation vector $\mathbf{k}$.

3. Thus $\begin{cases} \vec{S}_- = S(\omega,\gamma,B)\,\vec{S}_+ \\ \vec{S}^*_+ = \tilde{S}\,\vec{S}^*_- \end{cases} \Rightarrow \begin{cases} \vec{S}_- = S\,\vec{S}_+ \\ \tilde{S}^{-1}\vec{S}^*_+ = \vec{S}^*_- \Leftrightarrow \vec{S}_- = \tilde{S}^{-1*}\,\vec{S}_+ \end{cases}$

$$\Rightarrow (S - \tilde{S}^{-1*})\,\vec{S}_+ = \vec{0} \quad \forall\ \vec{S}_+ \Rightarrow S = \tilde{S}^{-1*} \Rightarrow \tilde{S} = S^{*-1}$$

$$\Rightarrow \boxed{\tilde{S} = S(\omega,-\gamma,-B) = [S^*(\omega,\gamma,B)]^{-1}}$$

This equation just describes how time-reversal transforms S.

Now, if a system has time-reversal symmetry then $S = \tilde{S}$

which implies $\begin{cases} S(\omega,\gamma,B) = S(\omega,-\gamma,-B) & (1) \\ S(\omega,\gamma,B) = [S^*(\omega,\gamma,B)]^{-1} & (2) \end{cases}$

AND

If we believe in the fact that S cannot have the same value at $+B$ and $-B$, and for $+\gamma$ and $-\gamma$, then (1) implies $\gamma = B = 0$.

(2) $\Rightarrow$ $S = S^{*-1}$ or $S^{-1} = S^*$

$$\boxed{\text{TRS} \Leftrightarrow \begin{array}{l} \gamma = B = 0 \\ S^{-1} = S^* \end{array}}$$

4. Reciprocity

4. If S is flux preserving and time reversal symmetric then

$$S^{-1} = S^{\dagger} = S^{*} \Rightarrow \underline{S = S^{T}}$$ reciprocity condition

⚠ this is only a sufficient condition: systems with $B = 0$ and $\gamma \neq 0$ are also reciprocal (see Lorentz reciprocity theorem, valid in the presence of losses). This is a consequence of the Onsager relation:

$$S(\omega, \gamma, -B) = S^{T}(\omega, \gamma, B)$$

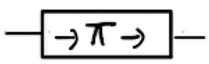
whose proof can be found in my Ph.D thesis, chap. 2 [UT Austin]

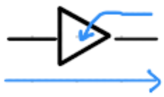
$S = S^{T} \Rightarrow B = 0$ but not $\gamma = 0$.

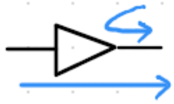
4. <u>example</u>: Can lossless 2 port systems be non-reciprocal ?

Lossless 2-port : $S = \begin{pmatrix} r e^{j\alpha} & t e^{j\beta} \\ -e^{j\varphi} t e^{-j\beta} & e^{j\varphi} r e^{-j\alpha} \end{pmatrix}$

Yes ! but only in phase (non-reciprocal phase shifters)

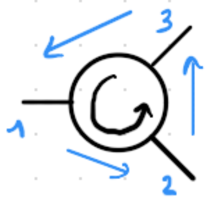
$\Rightarrow$ The gyrator  , $S = \begin{pmatrix} 0 & 1 \\ e^{j\pi} & 0 \end{pmatrix}$ is lossless

$\Rightarrow$ The isolator  , $S = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ must be lossy

 , $S = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ also !

What about 3 ports ?

$\Rightarrow$ The circulator $S = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is unitary non-reciprocal



Rq: If I match port 3, creating a 2 port system between 1 & 2, I get an isolator.

Is it unitary ? No, absorption occurs now at the matched port...

A few additional remarks (only for interested students!)

1. Another relation, called classical microreversibility, exists:

$$S(\omega, -\gamma, B) = \left[S^{+}(\omega, \gamma, B)\right]^{-1}$$

Proof: hard exercise!

2. Parity exchange left and right: $P\,S(\omega, \gamma(x), B(x)) = \sigma_x\, S(\omega, \gamma(-x), B(-x))\,\sigma_x$

assuming S is written as $S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$ where r, r', t, t' are blocks describing reflection & transmission when probed from left/right (dimensions $N \times N$) and $\sigma_x = \begin{pmatrix} 0 & \mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix}$

$N = \frac{m}{2}$ modes on each side in x direction.

3 Parity time acts on S as:

$$PT\, S(\omega, \gamma(x), B(x)) = \sigma_x \left[S^{*}(\omega, \gamma(-x), B(-x))\right]^{-1} \sigma_x$$

$$= \sigma_x\, S(\omega, -\gamma(-x), -B(-x))\, \sigma_x$$

PT symmetry implies $\gamma(x) = -\gamma(-x)$, $B(x) = -B(-x)$.

$$\sigma_x\, S\, \sigma_x = S$$

5. Scattering time

* <u>2 port case</u> : excitation from port 1

$$S_1^+ = e^{j(\omega t - kx)} \qquad S_2^+ = 0$$

$$S_1^- = r\, e^{j(\omega t - kx + \varphi_r)} \qquad S_2^- = t\, e^{j(\omega t - kx + \varphi_t)}$$

definition : $\boxed{\tau_r = \dfrac{\partial \varphi_r}{\partial \omega} \qquad \tau_t = \dfrac{\partial \varphi_t}{\partial \omega}}$ Wigner-Smith time delays (>0 or <0)

* <u>m-port generalisation</u>

Let's introduce the Wigner-Smith time delay operator for any unitary matrix S :

$$\boxed{Q_\omega = -j\, S^+ \frac{\partial S}{\partial \omega} = -j\, S^{-1} \frac{\partial S}{\partial \omega}}$$

<u>Properties</u> :

1) Q is Hermitian

2) It's eigenvectors are scattering states with well defined scattering times, given by the associated eigenvalues.

Proof:

1) $S^\dagger S = \mathbb{1} \Rightarrow \frac{\partial}{\partial \omega}(S^\dagger S) = 0 \Rightarrow \frac{\partial S^\dagger}{\partial \omega} S + S^\dagger \frac{\partial S}{\partial \omega} = 0$

$\Rightarrow \left(S^\dagger \frac{\partial S}{\partial \omega}\right)^\dagger + S^\dagger \frac{\partial S}{\partial \omega} = 0 \Rightarrow -j\, S^\dagger \frac{\partial S}{\partial \omega} = +j \left(S^\dagger \frac{\partial S}{\partial \omega}\right)^\dagger$

$\Rightarrow -j\, S^\dagger \frac{\partial S}{\partial \omega} = \left(-j\, S^\dagger \frac{\partial S}{\partial \omega}\right)^\dagger \Rightarrow Q_\omega = Q_\omega^\dagger$ QED

2) eigenvectors of Q_ω : $Q_\omega \vec{u}_m = \tau_m \vec{u}_m = -j\, S^{-1} \frac{\partial S}{\partial \omega} \vec{u}_m$

Noting $\vec{v}_m$ the output state corresponding to $\vec{u}_m$, $\vec{v}_m = S \vec{u}_m$ at a frequency ω_0

$+j\, \tau_m \vec{v}_m(\omega_0) = \left.\frac{\partial S}{\partial \omega}\right|_{\omega_0} \vec{u}_m(\omega_0) = \left.\frac{\partial \vec{v}_m(\omega_0)}{\partial \omega}\right|_{\omega_0} = +j\, \tau_m \vec{v}_m(\omega_0)$

$\Rightarrow \vec{v}_m(\omega_0 + \Delta\omega) \simeq \vec{v}_m(\omega_0)\, e^{+j \underbrace{\tau_m}_{\frac{d\varphi_{out}}{d\omega}} \Delta\omega}$

$\Rightarrow$ the scattering state is robust and its time delay is τ_m

6. Generalized Wigner-Smith operator

Let α be a parameter influencing the scattering process (ex: ω, as before, or the position x or angle θ of a given scatterer), and let $C_\alpha = -j\frac{d}{d\alpha}$ the conjugate operator to the observable α (related to the conjugated observable, ex t, p_x, L_x). $p_x \sim -j\frac{d}{dx}$

We define the generalized Wigner-Smith operator for unitary S matrices:

$$\boxed{Q_\alpha = -j\, S^{-1} \frac{\partial S}{\partial \alpha}}$$

Theorem: For an input state $\vec{u}$ associated with $\vec{v} = S\vec{u}$

$$\boxed{\vec{u}^{\dagger} Q_\alpha \vec{u} = \vec{u}^{\dagger} C_\alpha \vec{u} - \vec{v}^{\dagger} C_\alpha \vec{v} = \langle C_\alpha \rangle_{in} - \langle C_\alpha \rangle_{out}}$$

namely $\vec{u}^{\dagger} Q_\alpha \vec{u}$ measures the change of expectation value $\langle C_\alpha \rangle$ of the conjugate variable to α as the state $\vec{u}$ scatters into $\vec{v}$.

ex: $\alpha = x$, the position of a scatterer. $\vec{u}^{\dagger} Q_\alpha \vec{u}$ is the expected momentum shift that the eigenstate experiences, i.e the momentum transferred to the scatterer.

[see example discussed in class]

proof: We need to use shift operators. For a function f of $x \in \mathbb{R}$, there exists an operator T_t that shifts $f(x)$ by t namely $T_t f(x) = f(x+t)$.

Lemma: The expression of T_t is $T_t = e^{t\frac{d}{dx}}$, which means

$$T_t = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \frac{d^n}{dx^n}$$

[Proof of Lemma: Since all well-behaved functions can be expanded as $f(x) = \sum_{N=0}^{\infty} \frac{f^{(N)}(0)}{N!} x^N$, we can restrict the proof to $f = x^N$

$$T_t x^N = \sum_{n=0}^{+\infty} \frac{t^n}{n!} \frac{d^n}{dx^n} x^N = \sum_{n=0}^{N} \frac{t^n}{n!} \frac{N!}{(N-n)!} x^{N-n} = \sum_{n=0}^{N} \binom{N}{n} x^{N-n} t^n$$

$$= (x+t)^N \quad \text{(Binomial theorem)}$$]

f could be a complex vector like $\vec{u}$, T_t just applies to each components. For a function of α, $\boxed{T_{\Delta\alpha} = e^{\Delta\alpha \frac{d}{d\alpha}} = e^{j\Delta\alpha C_\alpha}}$

How to shift an operator O? We must shift all matrix elements

$$\langle \psi_j | O | \psi_i \rangle = \langle \psi_j(\alpha) | O | \psi_i(\alpha) \rangle \to \langle \psi_j(\alpha+\Delta\alpha) | O | \psi_i(\alpha+\Delta\alpha) \rangle$$

$$= \langle T_{\Delta\alpha} \psi_j | O | T_{\Delta\alpha} \psi_i \rangle = \langle \psi_j | T^{-1} O T | \psi_i \rangle$$

$$\Rightarrow \boxed{O(\alpha+\Delta\alpha) = T_{\Delta\alpha}^{-1} O(\alpha) T_{\Delta\alpha} = e^{-j\Delta\alpha C_\alpha} O \, e^{j\Delta\alpha C_\alpha}}$$

In particular, for S: $S(\alpha+\Delta\alpha) = e^{-j\Delta\alpha C_\alpha} S(\alpha) e^{j\Delta\alpha C_\alpha}$

If $\Delta\alpha$ is small $e^{\Delta\alpha \frac{d}{d\alpha}} \simeq \mathbb{1} + \Delta\alpha \frac{d}{d\alpha} = \mathbb{1} + j\Delta\alpha C_\alpha$

$$\Rightarrow S(\alpha+\Delta\alpha) \simeq (\mathbb{1} - j\Delta\alpha C_\alpha) S(\alpha) (\mathbb{1} + j\Delta\alpha C_\alpha)$$

$$\simeq S(\alpha) - j\Delta\alpha C_\alpha S_\alpha + j S(\alpha)\Delta\alpha C_\alpha + O(\Delta_\alpha^2)$$

$$\Rightarrow \frac{dS}{d\alpha} = \lim_{\alpha \to 0} \frac{S(\alpha+\Delta\alpha) - S(\alpha)}{\Delta\alpha} = j S(\alpha) C_\alpha - j C_\alpha S(\alpha)$$

$$\Rightarrow Q_\alpha = -j S^\dagger \frac{dS}{d\alpha} = -j S^\dagger (j S C_\alpha - C_\alpha S)$$

$$\Rightarrow \boxed{Q_\alpha = C_\alpha - S^\dagger C_\alpha S}$$

For an incident state $\vec{a}$, $\vec{b} = S\vec{a}$

$$\langle a|Q_\alpha|a\rangle = \vec{a}^\dagger Q \vec{a} = \vec{a}^\dagger C_\alpha \vec{a} - \underbrace{\vec{a}^\dagger S^\dagger}_{\vec{b}^\dagger} C_\alpha \underbrace{S\vec{a}}_{\vec{b}}$$

$$\Rightarrow \langle Q_\alpha \rangle_{in} = \langle C_\alpha \rangle_{in} - \langle C_\alpha \rangle_{out} \quad [QED]$$

This is in particular true for an eigenstate $\vec{u}$, $\vec{u}^\dagger Q_\alpha \vec{u} = \theta_\alpha$.

III) Free-space scattering

The impinging field does not match the BC on the obstacle surface

$$\Rightarrow \begin{cases} \vec{E}_{tot} = \vec{E}_{inc} + \vec{E}_{scat} \\ \vec{H}_{tot} = \vec{H}_{inc} + \vec{H}_{scat} \end{cases}$$

$$\Rightarrow \vec{S}_{tot} = \frac{1}{2}\vec{E}_{tot} \times \vec{H}_{tot}^{*} = \frac{1}{2}\left(\vec{E}_{inc} \times \vec{H}_{inc}^{*}\right) + \frac{1}{2}\left(\vec{E}_{scat} \times \vec{H}_{scat}^{*}\right) + \frac{1}{2}\left(\vec{E}_{inc} \times \vec{H}_{scat}^{*} + \vec{E}_{scat} \times \vec{H}_{inc}^{*}\right)$$

$$\Rightarrow Re\left[\oint d\vec{S} \cdot \vec{S}_{tot}\right] = \underbrace{Re\left[\frac{1}{2}\oint\left(\vec{E}_{tot} \times \vec{H}_{tot}^{*}\right) \cdot d\vec{S}\right]}_{W_{abs}} = \underbrace{Re\left[\frac{1}{2}\oint\left(\vec{E}_{inc} \times \vec{H}_{inc}^{*}\right) \cdot d\vec{S}\right]}_{0} +$$

$$\underbrace{Re\left[\frac{1}{2}\oint\left(\vec{E}_{scat} \times \vec{H}_{scat}\right) \cdot d\vec{S}\right]}_{W_{scat}} + \underbrace{Re\left[\frac{1}{2}\oint d\vec{S} \cdot \left(\vec{E}_{inc} \times \vec{H}_{scat} + \vec{E}_{scat} \times \vec{H}_{inc}^{*}\right)\right]}_{-W_{ext}}$$

$$\Rightarrow \boxed{W_{ext} = W_{scat} + W_{abs}}$$

extracted power from the incident field

= EXTINCTION POWER

* Remarques

- Lossless case : $W_{abs} = 0 \Rightarrow W_{ext} = W_{abs}$

- Scattering cross-section:

$$\sigma_{scat} = \frac{W_{scat}}{|S_{inc}|} = \frac{\text{total scattered power}}{\text{impinging power density}} \equiv \left[\frac{W}{W/m^2}\right] \equiv [m^2]$$

$$Q_{scat} = \frac{C_{scat}}{\pi a^2} \; [1] \; \text{scattering efficiency}$$

$$\sigma_{\substack{ext\\abs}} = \frac{W_{\substack{ext\\abs}}}{|S_{inc}|} \; [m^2]$$

$$\Rightarrow \quad \sigma_{ext} = \sigma_{scat} + \sigma_{abs}$$

- σ depends on the object's geometry